is calculated to be 3.6×10^{-5} cm/s, from the results of Hieber and Gebhart [8]. Since U at G_N^* is 0.2 cm/s, this disturbance level is only 0.018%. The disturbance level increases to 1.8% at $G^* = 440$, a 100-fold growth. The physical magnitude increases by about 450 fold.

We note that the velocity difference between two locations across the boundary layer around the inflexion point, and only 10^{-4} cm apart, is 2.4×10^{-5} cm/s at G_N^*. This places in proper perspective the very small magnitude of the actual physical disturbance, or mixing, which will induce nonlinear effects at $G^* = 440$ and cause transition at $G^* = 625$. A disturbance in the heat flux q'' of the order of 1% gives rise to the same disturbance magnitude. For even a much smaller disturbance, say one-tenth, transition will still correlate approximately to the same value of E. This suggests an experiment to determine G_{Tr}^* in a very quiet ambient with artificially induced disturbances, at various amplitudes.

3. CONCLUSIONS

For a naturally occurring input disturbance with no dominant components, the disturbance frequencies in actual transition are determined by frequency filtering. Variations in the input amplitude are estimated to have a small effect on the appearance of transition and on its progression. The initial input disturbance for observed transition is estimated and found to be very small. Flow interaction with background disturbances is very complex and future research is necessary to answer several remaining fundamental questions.

REFERENCES

1. B. Gebhart, Instability, transition, and turbulence in buoyancy-induced flows, *Ann. Rev. Fluid Mech.* **5**, 213 (1973).
2. B. Gebhart and R. Mahajan, Characteristic disturbance frequency in vertical natural convection flow, *Int. J. Heat Mass Transfer* **18**, 1143 (1975).
3. T. Audunson and B. Gebhart, Secondary mean motions arising in a buoyancy-induced flow, *Int. J. Heat Mass Transfer* **19**, 737 (1976).
4. Y. Jaluria and B. Gebhart, An experimental study of non-linear disturbance behavior in natural convection, *J. Fluid Mech.* **61**, 337 (1973).
5. F. Godaux and B. Gebhart, An experimental study of the transition of natural convection flow adjacent to a vertical surface, *Int. J. Heat Mass Transfer* **17**, 93 (1974).
6. Y. Jaluria and B. Gebhart, On transition mechanisms in vertical natural convection flow, *J. Fluid Mech.* **66**, 309 (1974).
7. L. Mahajan and B. Gebhart, An experimental investigation of transition in natural convection flows, submitted for publication.
8. C. A. Hieber and B. Gebhart, Stability of vertical natural convection boundary layers: some numerical solutions, *J. Fluid Mech.* **48**, 625 (1971).
9. Y. Jaluria and B. Gebhart, Stability and transition of buoyancy-induced flows in a stratified medium, *J. Fluid Mech.* **66**, 593 (1974).
10. P. S. Klebanoff, K. D. Tidstrom and L. M. Sargent, The three-dimensional nature of boundary-layer instability, *J. Fluid Mech.* **12**, 1 (1972).

Int. J. Heat Mass Transfer. Vol. 20, pp. 437–440. Pergamon Press 1977. Printed in Great Britain

GENERALIZED DIRECT EXCHANGE FACTORS FOR ISOTHERMAL MOLECULAR GASES

DUANE A. NELSON*
Department of Mechanical Engineering, The Pennsylvania State University, University Park, PA 16802, U.S.A.

(*Received* 5 *May* 1976 *and in revised form* 14 *September* 1976)

NOMENCLATURE

A, dimensionless total band absorption;
A_s, dimensionless slab band absorption;
d, diameter;
D, band width parameter;
E_n, exponential integral;
E_ν, Planck radiosity;
F, defined by equation (14);
Gs, gas–surface exchange factor [dimensionless];
h, slab thickness;
K_n, nongray transfer functions;
L, length;
L_e, mean beam length;
L_0, optically thin mean beam length;
L_∞, optically thick mean beam length;
q_e^r, emitted radiative flux;
q_a^r, absorbed radiative flux;
R, radius;
R_ν, spectral boundary radiosity;
S, integrated band intensity;
Sg, surface–gas exchange factor;
t, path length;
T_g, gas temperature;
u, dimensionless mean beam length.

Greek symbols
β, band fine-structure parameter;
μ, direction cosine;
ρ_a, absorbing gas density;
τ_x, optical depth based upon length x;
ϕ, azimuthal angle.

*Assistant Professor.

INTRODUCTION

ENGINEERING approximations for the analysis of radiative energy transfer from gases frequently involves the assumption of a uniform temperature. This proves to be a useful concept, for example, in a highly turbulent, well-stirred reactor and thus finds considerable application in the design of combustion devices for varied purposes. The isothermal assumption reduces the calculation of radiative transfer to the evaluation of transfer integrals depending only upon the geometry of the enclosure, with the radiation properties appearing parametrically. In general, closed form solutions are possible only for a limited number of simple configurations or in the limits of small and large optical paths. Hottel and Sarofim [1] have given a rather complete discussion of exchange areas and mean beam lengths for various geometries. Their discussion is largely limited to gray gases. These topics are also discussed from an occasionally different point of view by Siegel and Howell [2].

Calculations of exchange factors and mean beam lengths based upon nongray band absorption models are not especially abundant. Tien and Wang [3] have evaluated mean beam lengths for parallel plates, infinite cylinders and spheres based upon power law and logarithmic band absorption equations. Tien and Ling [4] have used a continuous, overlapped line, asymptotically logarithmic equation and have evaluated mean beam lengths for parallel and spherical geometries for a full range of optical paths. Edwards and Balakrishnan [5] have similarly evaluated mean beam lengths for the parallel geometry based upon a theoretical, overlapped line, exponential band model. Each of these investigations has been concerned with rather specific band absorption models of limited applicability. The development and utilization and band absorption equations is yet in a state of flux and there is a clear lack of agreement that any one formulation is preferable to another, since disagreement among them often appears more a matter of detail than of substance. In view of this it seems appropriate to develop generalized results which are valid for all band models of a given class. For molecular gases this can be accomplished in terms of the generalized transfer functions arising in the study of planar or spherical media.

FUNDAMENTALS

The radiative flux emitted by an isothermal gas volume to an area on its boundary may be expressed in terms of the dimensionless total band absorption as

$$q_e^r = \frac{DE_\nu(T_g)}{\pi}\int_0^{2\pi}\int_0^1 A[\tau_t(\mu,\phi)]\mu\,\mathrm{d}\mu\,\mathrm{d}\phi, \qquad (1)$$

where a single band has been assumed for simplicity, D is the band width parameter, $E_\nu(T_g)$ is the Planck radiosity evaluated at the band center or head and gas temperature, τ_t is the optical path through the gas for a given direction cosine, μ, and azimuthal angle, ϕ, and is given by $\tau_t = \rho_a St/D$ with ρ_a the absorbing medium density, S the integrated band intensity and t the physical path length. If the boundary of the gas is a diffuse black surface, equation (1) will give the boundary energy flux absorbed by the medium, q_a^r, when $E_\nu(T_g)$ is replaced by the boundary radiosity, R_ν. The dimensionless quantity $q_e^r/DE_\nu(T_g) = q_a^r/DR_\nu$ represents the exchange factor per unit area between the gas and the boundary, thus

$$Gs = Sg = \frac{1}{\pi}\int_0^{2\pi}\int_0^1 A[\tau_t(\mu,\phi)]\mu\,\mathrm{d}\mu\,\mathrm{d}\phi. \qquad (2)$$

This quantity is, of course, intimately connected with the direct-exchange-area of Hottel [1]. The present treatment will emphasize the nongray character of the medium.

Nongray transfer functions

Since the objective is to arrive at explicit expressions of equation (2) valid for arbitrary band absorption formulations, some consideration of the nongray transfer functions which arise in the study of planar media is needed. These functions have been defined by Nelson [6] in the form,

$$K_n(x) = (-)^n\int_0^1 A^{(3-n)}(x/\mu)\mu^{n-2}\,\mathrm{d}\mu$$
$$= (-)^n x^{n-1}\int_x^\infty A^{(3-n)}(\eta)\frac{\mathrm{d}\eta}{\eta^n} \qquad (3)$$

where

$$A^{(m)}(x) = \mathrm{d}^m A(x)/\mathrm{d}x^m; \quad m > 0$$
$$A^{(m)}(x) = \int_0^x \cdots \int_0^{y_2} A(y_1)\,\mathrm{d}y_1 \ldots \mathrm{d}y_{|m|}; \quad m < 0.$$

The $K_n(x)$ functions satisfy two recurrence relations [7],

$$\mathrm{d}K_{n+1}(x)/\mathrm{d}x = -K_n(x) \qquad (4)$$

and

$$nK_{n+1}(x) = (-)^{n+1}A^{(2-n)}(x) - xK_n(x), \qquad (5)$$

which will be useful in what follows. Explicit formulae for the $K_n(x)$ for various overlapped-line, band absorption models are available elsewhere [6, 8]. Although it has not been explicitly indicated, the foregoing relations also apply for band models depending upon a line structure parameter. In this regard, Lin and Chan [9] have presented results for the slab band absorptance [5] $A_s(x)$ and for $A_s'(x) = \mathrm{d}A_s(x)/\mathrm{d}x$ based upon the band absorption equations of Edwards and Menard [10] and Tien and Lowder [11]. In terms of the $K_n(x)$ functions these results are expressed by $A_s(x) = -2K_3(x)$ and $A_s'(x) = 2K_2(x)$. Nelson [7] has evaluated $K_2(x)$ for the band model due to Goody and Belton [12]. For other values of n, equation (5) proves to be quite useful.

APPLICATIONS

There appears to be only a few simple geometric configurations which allow an explicit evaluation of equation (2). Most of these have been discussed by Hottel and Sarofim [1] for a gray medium. All possess azimuthal symmetry and thus equation (2) reduces to,

$$Gs = Sg = 2\int_0^1 A[\tau_t(\mu)]\mu\,\mathrm{d}\mu. \qquad (6)$$

The dimensionless mean beam length u is defined by $A(xu) = Gs(x)$.

Planar medium

This geometry has been discussed in a somewhat different context by Edwards and Balakrishnan [5], who give results for the slab band absorption of an overlapped line exponential band, by Nelson [6] who studied the $K_n(x)$ functions for several overlapped line band absorption models and by Lin and Chan [9] who obtained the expression for slab band absorption based on the Edwards and Menard [10] and Tien and Lowder [11] equations, thus, including line structure effects. For the planar geometry the argument of $A(x)$ in equation (6) is simply $\tau_t(\mu) = \tau_h/\mu$ where $\tau_h = \rho_a Sh/D$ and h is the plate spacing or slab thickness. Substitution of this into equation (6) and reference to equation (3) with $n = 3$ shows that,

$$Gs = -2K_3[\tau_h]. \qquad (7)$$

This, of course, is the same as slab band absorption but in this form is valid for all band absorption models. In the general case the mean beam length for the planar geometry, $u = L_e/h$, is obtained by solving the transcendental equation

$$A(\tau_h u) = -2K_3(\tau_h). \qquad (8)$$

This calculation has been performed by Edwards and Balakrishnan [5] for the overlapped line, exponential band model and by Tien and Ling [4] for another overlapped line absorption model. The effect of line structure is not often considered but could easily be accommodated by equation (8). For example, in the nonoverlapped strong-line limit $A(x) = 2\sqrt{(\beta x)}$, $-2K_3(x) = (8/3)\sqrt{(\beta x)}$ and $u = 16/9$ as found by Elsasser [13] for his narrow band model.

Circular cylinder radiating to center of one end

Another geometry which allows evaluation of equation (6) is a circular cylinder of radius R and length L when the receiving area is a differential spot at the center of one end. In this case the form of $\tau_t(\mu)$ depends upon whether the direction cosine causes the beam to intersect the opposite end or the side of the cylinder. Consequently the integral in equation (6) must be split into two parts so that,

$$Gs = 2\int_0^{L/(R^2+L^2)^{1/2}} A[\tau_R/(1-\mu^2)^{1/2}]\mu\,\mathrm{d}\mu + 2\int_{L/(R^2+L^2)^{1/2}}^1 A[\tau_L/\mu]\mu\,\mathrm{d}\mu. \qquad (9)$$

Introducing a change of variable, $\mu' = (1-\mu^2)^{1/2}$, into the first term and using some subsequent rearrangement gives,

$$Gs = 2K_3[\tau_R(1+L^2/R^2)^{1/2}] - 2K_3[\tau_R] - 2K_3[\tau_L]. \quad (10)$$

As would be expected, as $L/R \to 0$ $Gs \to -2K_3[\tau_L]$ so that the behavior is like that of planar medium of thickness L. It is an interesting result that as $L/R \to \infty$ $Gs \to -2K_3[\tau_R]$, thus, as is known, a semi-infinite cylinder radiating to the center of its base produces the same flux as an infinite slab whose thickness is equal to the radius of the cylinder. Insofar as mean beam length is concerned it is obvious that the behavior of the two limits discussed above would parallel that of a slab. For intermediate conditions it is convenient to consider a cylinder with $L = 2R$ for the overlapped-line, exponential band model. In the optically thin limit $-2K_3(x) \to 2x$ while $A(x) \to x$; thus, putting $A(\tau_d u) = Gs(\tau_d)$ with $\tau_d \to 0$ gives $u_0 = 3-\sqrt{5} = 0.7639$ which agrees with the result given by Hottel and Sarofim [1], as it must. Letting $\tau_d \to \infty$ results in $-2K_3(\tau_d) \sim \ln\tau_d + \gamma + 1/2$ and $A(\tau_d) \sim \ln\tau_d + \gamma$. With these one finds $u_\infty = (e/5)^{1/2} = 0.7373$ and thus $u_\infty/u_0 = 0.965$.

For other values of L/R one finds in the optically thin limit that $L_0 = 2[R+L-(R^2+L^2)^{1/2}]$ while the optically thick exponential band yields $L_\infty = e^{1/2}L/(1+L^2/R^2)^{1/2}$. The ratio u_∞/u_0 is expressed as,

$$\frac{u_\infty}{u_0} = \frac{e^{1/2}}{2}\frac{X}{(1+X^2)^{1/2}[1+X-(1+X^2)^{1/2}]} \quad (10a)$$

where $X = L/R$. This expression has a broad maximum about $X = 1$ where $u_\infty/u_0 = 0.9951$. Since one expects the mean beam length to be monotonic in optical depth this represents a remarkably small variation which may have possible applications.

The sphere

Although the spherical medium is only a special case of the spherical shell considered next, it is convenient to treat it separately. In this case it is easily shown that $\tau_t(\mu) = \tau_d\mu$ and equation (6) becomes

$$Gs = 2\int_0^1 A[\tau_d\mu]\mu\,d\mu. \quad (11)$$

Since the integrand does not correspond to any of the $K_n(x)$ functions it is necessary here to use equation (5) with $n = 2$, and then to apply equation (4) with the result,

$$Gs = \frac{2}{\tau_d^2}\int_0^{\tau_d}\left[2\frac{dK_4(\eta)}{d\eta} + \eta\frac{dK_3(\eta)}{d\eta}\right]\eta\,d\eta. \quad (12)$$

A straightforward integration by parts using equation (4) and $K_n(0) = 0$ for $n > 2$ yields

$$Gs = F[\tau_d], \quad (13)$$

with

$$F[\tau_d] = 2K_3[\tau_d] + 8K_4[\tau_d]/\tau_d + 8K_5[\tau_d]/\tau_d^2. \quad (14)$$

Explicit formulae for the $K_n(x)$ up to $n = 5$ have been given by Crosbie and Khalil [8] for gray, triangular and exponential bands with overlapped lines. Nelson [6] has obtained results up to $n = 4$ for several logarithmic type absorption models with overlapped lines. For these later cases the results for $K_5(x)$ can be easily obtained by application of equation (5). Expressions with $n > 3$ for band models with line structure effects have not been reported, but again application of equation (5) can be made. If one substitutes the expressions for a gray band into equation (14) the result given by Hottel and Sarofim ([1], p. 267) is recovered. For an overlapped-line exponential band equation (14) yields,

$$Gs = \gamma - 1/2 + \ln\tau_d - 2E_2[\tau_d]/\tau_d + \{1-2E_3[\tau_d]\}/\tau_d^2. \quad (15)$$

The asymptotic behavior of the mean beam length has been discussed by Tien and Wang [3]. The results are $u_0 = 2/3$, $u_\infty = e^{-1/2} = 0.6065$ and $u_\infty/u_0 = 0.9098$.

The spherical shell

In the spherical shell the gas is confined between spheres of radius R_1 and R_2 where $R_1 < R_2$. The exchange factor between the gas and the inner surface is given by the integral expression,

$$Gs_1 = 2(R_2/R_1)^2\int_{[1-(R_1/R_2)^2]^{1/2}/2}^{[1-(R_1/R_2)]} A(\tau_{d_2}\eta) \times \left\{\eta - \frac{[1-(R_1/R_2)^2]^2}{16\eta^3}\right\}d\eta \quad (16)$$

which yields,

$$Gs_1 = \frac{(R_2/R_1)^2}{2}\{2F[\tau_{d_2}(1-R_1/R_2)/2] - 2F[\tau_{d_2}\{1-(R_1/R_2)^2\}^{1/2}/2] + [1-(R_1/R_2)^2]K_3[\tau_{d_2}\{1-(R_1/R_2)^2\}^{1/2}/2] - [1+R_1/R_2]^2K_3[\tau_{d_2}(1-R_1/R_2)/2]\}. \quad (17)$$

The exchange between the gas and the outer surface results in the integral expression,

$$Gs_2 = 2\int_0^{[1-(R_1/R_2)^2]^{1/2}} A(\tau_{d_2}\eta)\eta\,d\eta + (R_1/R_2)^2 Gs_1 \quad (18)$$

giving

$$Gs_2 = F[\tau_{d_2}\{1-(R_1/R_2)^2\}^{1/2}] + (R_1/R_2)^2 Gs_1. \quad (19)$$

When $R_1 = 0$ equation (19) reduces to equation (13) for a sphere.

The behavior of the mean beam length ratio, u/u_0, for exchange with the entire surface will be intermediate between that for the sphere and the planar medium for radius ratios in the range $0 < R_1/R_2 < 1$ where the characteristic beam length is based upon $(R_2 - R_1)$.

DISCUSSION

Each of the generalized exchange factors obtained in previous sections has the advantage of being valid for any molecular-gas band absorption model. Results which have been previously obtained for these configurations have for the most part been based upon either the gray band approximation or a specific nongray equation. Although Hottel and Sarofim ([1], p. 287) suggest that only gray results are needed, there have been significant improvements in the intervening years in the calculation methods for radiation properties of isothermal gases [14] which fully justify and indeed clearly indicate the need for nongray, preferably generalized, results. The weighted sum of gray gases procedure advocated by Hottel and Sarofim and others is totally dependent upon the availability of measured gas emissivities and even then is rather awkward to apply. In many cases the necessary results are lacking, especially for mixtures of radiating species. The lack of mixture data is a significantly lesser difficulty when nongray band absorption models are used [14]. When the bands are non-overlapping the calculation of radiative exchange for multiband media is straightforward. If overlapping is important, various procedures may be employed [14, 15]. In either case, nongray exchange factors are needed.

There is an interesting observation in connection with the present results and those for gray media. It has been found that if analytical results for a gray medium are available, these can be transformed into generalized results by relatively simple substitutions and algebraic manipulations, thus bypassing the need for evaluation of the integral in equation (6) in generalized terms. All that is needed is a table of transfer functions, $K_n(x)$, for a gray band medium. As an example, equation (10) can be obtained directly from gray results of [1], p. 267 by the substitution $1-2E_3(x) = -2K_3(x)$. Several other results given in [1] can be generalized in this way.

REFERENCES

1. H. C. Hottel and A. F. Sarofim, *Radiative Transfer*. McGraw-Hill, New York (1967).
2. R. Siegel and J. R. Howell, *Thermal Radiation Heat Transfer*. McGraw-Hill, New York (1972).
3. C. L. Tien and L. S. Wang, On the calculation of mean beam length for a radiating gas, *J. Quantve Spectrosc. Radiat. Transfer* **5**, 453–456 (1965).
4. C. L. Tien and G. R. Ling, On a simple correlation for total band absorptance of radiating gases, *Int. J. Heat Mass Transfer* **12**, 1179–1181 (1969).
5. D. K. Edwards and A. Balakrishnan, Slab band absorptance for molecular gas radiation, *J. Quantve Spectrosc. Radiat. Transfer* **12**, 1379–1387 (1972).
6. D. A. Nelson, A study of band absorption equations for infrared radiative transfer in gases—I. Transmission and absorption functions for planar media, *J. Quantve Spectrosc. Radiat. Transfer* **14**, 69–80 (1974).
7. D. A. Nelson, A recurrence relation for transmission and absorption functions of infrared radiating gases, *J. Quantve Spectrosc. Radiat. Transfer* **16**, 321–323 (1976).
8. A. L. Crosbie and H. K. Khalil, Mathematical properties of the $K_n(\tau)$ functions, *J. Quantve Spectrosc. Radiat. Transfer* **12**, 1457–1464 (1972).
9. C. C. Lin and S. H. Chan, A general slab band absorptance for infrared radiating gases, *J. Heat Transfer* **97**, 478–480 (1975).
10. D. K. Edwards and W. A. Menard, Comparison of models for correlation of total band absorption, *Appl. Optics* **3**, 621–625 (1964).
11. C. L. Tien and J. E. Lowder, A correlation for total band absorptance of radiating gases, *Int. J. Heat Mass Transfer* **9**, 698–701 (1965).
12. R. Goody and M. J. S. Belton, Radiative relaxation times for Mars, *Planet. Space Sci.* **15**, 247–256 (1967).
13. W. M. Elsasser, Heat transfer by infrared radiation in the atmosphere, *Harvard Meteor. Studies* **6** (1942).
14. D. K. Edwards and A. Balakrishnan, Thermal radiation by combustion gases, *Int. J. Heat Mass Transfer* **16**, 25–40 (1973).
15. S. S. Penner and P. Varanasi, Effects of (partial) overlapping of spectral lines on the total emissivity of H_2O–CO_2 mixtures, *J. Quantve Spectrosc. Radiat. Transfer* **6**, 181–192 (1966).